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EXACT CONFIDENCE COEFFICIENTS OF CONFIDENCE INTERVALS FOR A BINOMIAL PROPORTION

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Abstract: Let X have a binomial distribution $B(n,p)$. For a confidence interval $(L(X),U(X))$ of a binomial proportion p, the coverage probability is a variable function of p. The confidence coefficient of the confidence interval is the infimum of the coverage probabilities, $\inf_{0\le p\le 1} P_p(p \in (L(X),U(X)))$. Usually, the exact confidence coefficient is unknown since the infimum of the coverage probabilities may occur at any point $p \in (0,1)$. In this paper, a methodology to compute the exact confidence coefficient is proposed. With this methodology, the point where the infimum of the coverage probabilities occurs, as well as the confidence coefficient, can be precisely derived.

Key words and phrases: Binomial distribution, confidence coefficient, confidence interval, coverage probability.

1. Introduction

Let X have a binomial distribution $B(n,p)$. For a confidence interval $(L(X), U(X))$ of a binomial proportion p, the coverage probability of $(L(X),U(X))$ is the probability that the random interval $(L(X),U(X))$ covers the true parameter p. The coverage function is a variable function of p. The confidence coefficient of $(L(X),U(X))$ is the infimum of the coverage probabilities, $\inf_{0\le p\le 1} P_p(p \in (L(X),U(X)))$. Usually, the exact confidence coefficient is unknown since the infimum of the coverage probabilities may occur at any point $p \in (0,1)$. When n is large enough, the nominal coefficient is used to approximate it. However, in the real application of n being fixed, the confidence coefficient may be far below the nominal coefficient unless n is large. For example, consider the Wald confidence interval $\hat{p}\pm z_{\alpha/2}\sqrt{\hat{p}(1-\hat{p})/n}$, where $\hat{p}=X/n$. It is known that the coverage probability of the Wald interval tends to zero as p goes to 0 (see e.g., Blyth and Still (1983)). For another confidence interval $\hat{p} \pm z_{\alpha/2}\sqrt{p(1-p)/n}$, Huwang (1995) shows that the coverage probability is much smaller than the nominal coefficient. In this paper, a methodology to compute the exact confidence coefficient is proposed. It is shown that for a confidence interval $(L(X),U(X))$ satisfying some assumptions, the infimum coverage probability occurs at the point p being one of the endpoints $\{L(1),\ldots,L(n),U(0),\ldots,U(n-1)\}$.

The paper is organized as follows. In Section 2, the methodology of calculating the confidence coefficient is proposed, as well as a method to obtain the maximum coverage probability. In Section 3, by applying our procedure, the confidence coefficients for some interval estimations of a binomial proportion p recommended in Brown, Cai and DasGupta (2002) are derived.

2. Exact confidence coefficients

In this section, we propose a method of deriving the confidence coefficients for the confidence intervals satisfying Assumption 1.

Assumption 1. *A confidence interval* $(L(X), U(X))$ *of a binomial proportion* p *satisfies*

(i) $L(X_1) < L(X_2)$ *and* $U(X_1) < U(X_2)$ *if* $X_1 < X_2$*;*

(ii) *for any fixed* $p \in (0,1)$*, there exists* x_0 *such that* $p \in (L(x_0), U(x_0))$*.*

Condition (ii) in Assumption 1 guarantees that there does not exist a $p \in (0,1)$ such that the coverage probability at p is zero.

Before proceeding to the proof of the main result, we need the following definitions and lemma. For a confidence interval $(L(X), U(X))$, there are $2(n+1)$ endpoints, $L(0), \ldots, L(n), U(0), \ldots U(n)$, corresponding to $X = 0, \ldots, n$. We rank the $2(n+1)$ endpoints from the smallest to the largest. Let v_i, $i = 1, \ldots, 2n+2$, denote the *ith* smallest value of these endpoints. For a $p \in (v_i, v_{i+1})$, let $k_0(p)$ denote the smallest x such that $p < U(x)$ and $k_1(p)$ denote the largest x such that $L(x) < p$. By (i) in Assumption 1, the coverage probability of the interval at a parameter p is

$$P_p\Big(p \in (L(X), U(X))\Big) = \sum_{i=k_0(p)}^{k_1(p)} \binom{n}{i} p^i (1-p)^{n-i}. \tag{1}$$

Lemma 1.

(i) *For a fixed* m_1 *with* $0 < m_1 < n$, $\sum_{i=0}^{m_1} \binom{n}{i} p^i (1-p)^{n-i}$ *is a decreasing function of* p*.*

(ii) *For a fixed* m_0 *with* $0 < m_0 < n$, $\sum_{i=m_0}^{n} \binom{n}{i} p^i (1-p)^{n-i}$ *is an increasing function of* p*.*

(iii) *For a fixed* m_0 *and* m_1 *with* $0 < m_0 < m_1 < n$*, the function* $\sum_{i=m_0}^{m_1} \binom{n}{i} p^i (1-p)^{n-i}$ *is a unimodal function. The maximum of the function occurs at* $p = r(m_0, m_1)$*, where* $r(m_0, m_1) = 1/(1+M)$ *and*

$$M = \left(\frac{\binom{n}{m_1}(n-m_1)}{\binom{n}{m_0} m_0} \right)^{\frac{1}{(m_1 - m_0 + 1)}}.$$

The proof of Lemma 1 is in the Appendix.

Theorem 1. *For a confidence interval* $(L(X), U(X))$ *satisfying Assumption* 1, *the infimum coverage probability, confidence coefficient, is the minimum value of* $2n$ *coverage probabilities,*

$$\Big\{P_p(p \in (L(X), U(X))) : p \in S = \{L(1), \cdots, L(n), U(0), \cdots, U(n-1)\}\Big\}.$$

If $L(X_0) \leq 0$ *or* $U(X_0) \geq 1$ *for some* X_0, *then* S *is reduced to the set which excludes these points.*

Proof. Let

$$w_i = \begin{cases} v_i & \text{if } \ 0 < v_i < 1 \\ 0 & \text{if } \ v_i \leq 0 \\ 1 & \text{otherwise.} \end{cases} \tag{2}$$

We show that

$$\inf_{w_i < p < w_{i+1}} P_p\Big(p \in (L(X), U(X))\Big) = \min_{p = w_i \ or \ w_{i+1}} P_p\Big(p \in (L(X), U(X))\Big). \tag{3}$$

With this result, the infimum of the coverage probabilities for $p \in (w_i, w_{i+1})$ is the minimum of the coverage probabilities $P_{w_i}(w_i \in (L(X), U(X)))$ and $P_{w_{i+1}}(w_{i+1} \in (L(X), U(X)))$. Thus, when (3) holds, the infimum of the coverage probabilities for $p \in (0, 1)$ is the minimum of the $2n$ coverage probabilities $\{P_p(p \in (L(X), U(X))) : p \in S\}$. The reason that S excludes the points not larger than zero and the points not smaller than 1 is due to the parameter space being $(0, 1)$. According to Assumption 1 and Lemma 1, S does not contain $L(0)$ and $U(n)$.

Now we have to prove (3). Note that for a fixed i, $k_0(p)$ and $k_1(p)$ are constants for all $p \in (w_i, w_{i+1})$, because there are not any endpoints between w_i and w_{i+1}. According to Lemma 1, (1) is a unimodal function, a decreasing function or an increasing function of p. Thus, by the properties of these functions, the minimum value of $\inf_{w_i < p < w_{i+1}} P_p(p \in (L(X), U(X)))$ occurs at $p = w_i$ or w_{i+1}. Thus, the proof is complete.

Remark 1. Note that the confidence interval $(L(X), U(X))$ does not contain the point $L(X)$ or $U(X)$. When we count the intervals covering $p = L(X)$ or $p = U(X)$, we do not count the interval $(L(X), U(X))$.

According to Theorem 1, the following steps provide the exact confidence coefficient of a confidence interval $(L(X), U(X))$.

Procedure to compute confidence coefficients

Step 1. Check if the union of $(n + 1)$ intervals, $(L(0), U(0)), \ldots, (L(n), U(n))$, covers the interval $(0, 1)$ and if (i) in Assumption 1 is satisfied. If it does not cover $(0, 1)$, then the confidence coefficient is zero and we stop.

Step 2. Calculate the coverage probabilities of the confidence interval for p being the endpoints larger than zero and smaller than one. The minimum value of these coverage probabilities is the exact confidence coefficient.

In Theorem 2, a methodology for computing the maximum coverage probability is provided.

Theorem 2. *For a confidence interval* $(L(X), U(X))$ *satisfing Assumption* 1*, the maximum coverage probability of* $(L(X), U(X))$ *is the maximum value of the coverage probabilities,*

$$\{P_p(p \in (L(X), U(X))) : p \in S_1 = \{L(0), \ldots, L(n), U(0), \ldots, U(n), a_1, \ldots, a_{2n+1}\}\},$$

where a_i *is* $r(k_0((w_i + w_{i+1})/2), k_1((w_i + w_{i+1})/2))$.

Proof. Note that

$$\max_{0<p<1} P_p(p \in (L(X), U(X))) = \max_{i=1,\ldots,2n+1} \Big(\max_{v_i<p<v_{i+1}} P_p(p \in (L(X), U(X))) \Big). \quad (4)$$

Since (1) is a unimodal function, a decreasing function or an increasing function, $\max_{w_i<p<w_{i+1}} P_p(p \in (L(X), U(X)))$ occurs at $p = w_i, w_{i+1}$ or a_i. Hence, the maximum value of $P_p(p \in (L(X), U(X))$ occurs at a point in S_1.

3. Examples

In this section, we use the result in Section 2 to calculate the exact confidence coefficients of the four proposed confidence intervals in Brown et al. (2002). Note that the confidence intervals considered in Brown et al. (2002) are closed intervals, while we consider open intervals. Since the confidence coefficient is the infimum of the coverage probabilities, it is the same for both open and closed intervals. Let k be the upper $\alpha/2$ cutoff point of the standard normal distribution. The four intervals are introduced below.

1. The Wilson interval. Let $\tilde{X} = X + k^2/2$ and $\tilde{n} = n + k^2$. Let $\tilde{p} = \tilde{X}/\tilde{n}$, $\tilde{q} = 1 - \tilde{p}$, $\hat{p} = X/n$ and $\hat{q} = 1 - \hat{p}$. The $1 - \alpha$ Wilson interval has the form

$$CI_W(X) = \left(\tilde{p} - \frac{kn^{\frac{1}{2}}}{n + k^2} \Big(\hat{p}\hat{q} + \frac{k^2}{4n} \Big)^{\frac{1}{2}}, \quad \tilde{p} + \frac{kn^{\frac{1}{2}}}{n + k^2} \Big(\hat{p}\hat{q} + \frac{k^2}{4n} \Big)^{\frac{1}{2}} \right).$$

2. The Agresti-Coull interval. The $1 - \alpha$ Agresti-Coull interval is

$$CI_{AC}(X) = \Big(\tilde{p} - k(\tilde{p}\tilde{q})^{\frac{1}{2}} \tilde{n}^{-\frac{1}{2}}, \quad \tilde{p} + k(\tilde{p}\tilde{q})^{\frac{1}{2}} \tilde{n}^{-\frac{1}{2}} \Big).$$

3. The likelihood ratio interval. The $1 - \alpha$ interval is

$$CI_{\Lambda_n}(X) = \left\{ p : \frac{p^X (1-p)^{n-X}}{(\frac{X}{n})^X (1 - \frac{X}{n})^{n-X}} > e^{-\frac{k^2}{2}} \right\}.$$

4. The equal-tailed Jeffreys interval. The $1-\alpha$ equal-tailed Jeffreys prior interval is

$$CI_J(X) = \left(\beta_{\frac{\alpha}{2},X+\frac{1}{2},n-X+\frac{1}{2}}, \beta_{1-\frac{\alpha}{2},X+\frac{1}{2},n-X+\frac{1}{2}}\right),$$

where $\beta(\alpha, s_1, s_2)$ denotes the α quantile of a $Beta(s_1, s_2)$ distribution.

First, we follow the procedure in Section 2 to compute the confidence coefficients of the Wilson interval.

Example 1. For $n = 5$ and $k = 1.96$, the $CI_W(X)$ intervals corresponding to $X = 0, \ldots, 5$ are listed as follows: (0, 0.4345), (0.0362, 0.6245), (0.1176, 0.7693), (0.2307, 0.8824), (0.3755, 0.9638), (0.5655, 1). From these six intervals, it is clear that the two conditions in Assumption 1 are satisfied. The coverage probabilities are 0.9266, 0.8316, 0.9304, 0.8914, 0.9157, 0.9157, 0.8914, 0.9304, 0.8316, 0.9266, corresponding to the endpoints 0.4345, 0.0362, 0.6245, 0.1176, 0.7693, 0.2307, 0.8824, 0.3755, 0.9638, 0.5655. The minimum value of these coverage probabilities is 0.8316, which is the coverage probability of the Wilson interval at $p = 0.0362$ and $p = 0.9638$. Thus, the exact confidence coefficient is 0.8316.

Table 1 and Table 2 list the confidence coefficients of the Wilson interval and the Agresti-Coull interval corresponding to different n.

The two confidence intervals we discussed above have closed forms for the endpoints. However, the likelihood ratio interval does not have a closed form. Nevertheless, from the structure of the interval, $p^X(1-p)^{n-X}$ is a unimodal function of p. Hence, we can have the two endpoints of the intervals for each fixed x by numerical calculation. Table 3 lists the confidence coefficients corresponding to different n.

Table 1. The exact confidence coefficients of the 95% Wilson confidence intervals corresponding to different n.

n	Confidence Coefficient	p such that $P_p(p \in CI_W(X))$ =confidence coefficient
5	0.8315	0.0362,0.9638
20	0.8366	0.0088,0.9912
30	0.8371	0.0059,0.9941
50	0.8376	0.0035,0.9965
70	0.8377	0.0025,0.9975
90	0.8378	0.0019,0.9981
100	0.8379	0.0017,0.9983
300	0.8381	0.0005,0.9995
600	0.8381	0.0002,0.9998
900	0.8381	0.0001,0.9999

Table 2. The exact confidence coefficients of the 95% Agresti-Coull confidence intervals corresponding to different n.

n	Confidence Coefficient	p such that $P_p(p \in CI_{AC}(X))$ =confidence coefficient
5	0.8941	0.1159,0.8841
20	0.9292	0.4787,0.5213
30	0.9338	0.4868,0.5132
50	0.9345	0.3115,0.6885
70	0.9368	0.2296,0.7704
90	0.9377	0.3670,0.6330
100	0.9380	0.2454,0.7546
300	0.9416	0.0934,0.9066
600	0.9427	0.0308,0.9692
900	0.9433	0.0177,0.9823

Table 3. The exact confidence coefficients of the 95% likelihood ratio confidence intervals corresponding to different n.

n	Confidence Coefficient	p such that $P_p(p \in CI_{\Lambda_n}(X))$ =confidence coefficient
5	0.8150	0.3190,0.6810
20	0.8225	0.0916,0.9084
30	0.8178	0.0620,0.9380
50	0.8426	0.0377,0.9623
70	0.8420	0.0271,0.9729
90	0.8411	0.0211,0.9789
100	0.8408	0.0190,0.9810
300	0.8408	0.0064,0.9936
600	0.8402	0.0032,0.9968
900	0.8364	0.0021,0.9979

Consider the equal-tailed Jeffreys interval. Since the lower endpoints of the intervals are greater than zero for all $x = 0, \ldots, n$, there exists a $p_0 \in (0, 1)$ such that $p_0 < \min_{x=0,\ldots,n}(\beta_{\alpha/2,x+1/2,n-x+1/2})$, which implies that the confidence intervals do not cover p_0. Hence, the confidence coefficient of the equal-tailed Jeffreys interval is zero for all fixed n.

From Tables 1−3, the confidence coefficients of the Agresti-Coull intervals are higher than the other two intervals IC_W and IC_{Λ_n} because the Agresti-Coull intervals are substantially longer. The asymptotic expansions of expected lengths of the four confidence intervals and the Wald interval are given in Brown et al. (2002), see their Figure 9.

It is interesting to see that the confidence coefficients of IC_W and IC_{AC} have an increasing tendency in n, although not always increasing in n. This phenomenon does not occur in the confidence coefficients of IC_{Λ_n}.

The programs for computing the confidence coefficients of CI_W, CI_{AC} and CI_{Λ_n} are available on the website http://www.stat.sinica.edu.tw/hywang/programs.

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Appendix

Proof of Lemma 1 The binomial family has a monotone likelihood ratio in X. Then, by Lemma 2 of Chapter 3 in Lehmann (1986), for $p < p'$, the cumulative distribution functions of X under p and p' satisfy $F_p(x) > F_{p'}(x)$ for all x, which implies (i) and (ii). The unimodality property of (iii) also can be shown through an application of Theorem 6 of Chapter 3 in Lehmann (1986).

But, since the point where the maximum of the function occurs needed to be specified in Theorem 2, (iii) is also demonstrated by the following calculation.

Let $y(p) = \sum_{i=m_0}^{m_1} \binom{n}{i} p^i (1-p)^{n-i}$. Then

$$\begin{aligned}\frac{\partial y(p)}{\partial p} &= \sum_{i=m_0}^{m_1} \left[\binom{n}{i} i p^{i-1} (1-p)^{n-i} - \binom{n}{i} (n-i) p^i (1-p)^{n-i-1} \right] \qquad (5)\\ &= \binom{n}{m_0} m_0 p^{m_0-1} (1-p)^{n-m_0} - \binom{n}{m_1} (n-m_1) p^{m_1} (1-p)^{n-m_1-1} \\ &= p^{m_0-1}(1-p)^{n-m_1-1} \left(\binom{n}{m_0} m_0 (1-p)^{m_1+1-m_0} - \binom{n}{m_1} (n-m_1) p^{m_1+1-m_0} \right).\end{aligned}$$

We see that (5) has three roots $p = 0, p = 1$ and $p = r(m_0, m_1)$, where $r(m_0, m_1) = 1/(1+M)$ and

$$M = \left(\frac{\binom{n}{m_1}(n-m_1)}{\binom{n}{m_0} m_0} \right)^{\frac{1}{(m_1-m_0+1)}}.$$

Note that (5) is greater than zero if p is less than $r(m_0, m_1)$, and (5) is less than zero if p is larger than $r(m_0, m_1)$. Thus, $y(p)$ is a unimodal function with a maximum at $p = r(m_0, m_1)$.

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