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Modified p-values for one-sided testing in restricted parameter spaces

Hsiuying Wang

Institute of Statistical Science, Academia Sinica, Taipei 115, Taiwan

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Abstract

For testing the mean of a normal distribution, the p-value, derived from the uniformly most powerful test, is usually used as evidence against the null hypothesis. However, the p-value only depends on the hypothesis assumption, but not on the bounds of the parameter space. When the parameter space is restricted, the information of the restriction will not be sufficiently utilized if we still use the usual p-value as evidence against the null hypothesis. In this paper, a modified p-value, based on the bounds of the parameter space for one-sided hypothesis testing, is proposed. Theoretical and simulation studies show that the modified p-value has better performance than the usual p-value from theoretical and simulation studies.
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1. Introduction

Let X be a normal random variable with mean θ and variance σ^2, where θ is an unknown parameter and σ^2 is known. Without loss of generality, σ^2 is assumed to be 1 throughout the paper. For testing the hypothesis:

$$H_0 : \theta \in \Theta_0 = (-\infty, \theta_0] \quad \text{versus} \quad H_1 : \theta \in \Theta_1 = (\theta_0, \infty), \tag{1}$$

the suggestion of reporting a p-value derived from uniformly most powerful test as evidence against the null hypothesis is commonly accepted. Note that the p-value depends only on observation x and θ_0. When the parameter space $\Theta_0 \cup \Theta_1$ is a restricted space $(b, a), -\infty \leqslant b \leqslant a \leqslant \infty$, instead of the natural parameter space $(-\infty, \infty)$, we investigate whether the usual p-value, which does not depend on the bounds a and b, is still good evidence against the null hypothesis.

For the Poisson distribution, Woodroofe and Wang (2000) discuss this problem and point out the drawbacks of using the usual p-value as a measure of evidence. A modified p-value is proposed in their paper for this case. In the present paper, we propose a modified p-value for one-sided testing of the location of the normal distribution.

E-mail address: hywang@stat.sinica.edu.tw.

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The disadvantage of using the usual p-value as a measure of evidence for restricted parameter spaces is due to the fact that the information on the parameter space is not sufficiently utilized. Thus, we propose a modified p-value depending on the bounds of the parameter space. This modified p-value is shown to be a Bayes estimator in the terminology of Hwang et al. (1992), from which it follows that it is an admissible estimator of the indicator function of Θ_0. The usual p-value is shown to be inadmissible.

In many practical applications, the parameter space is restricted and the bounds are known. This bounded parameter space problem has been discussed in the recent literature. Mandelkern (2002) gives the examples where the classical Neyman procedure is not satisfactory to many scientists, when the parameter is known to be bounded. This problem occurs frequently in analyzing data from physics experiments (see Feldman and Cousins, 1998; Roe and Woodroofe, 2001). Moreover, many other criticisms for the usual p-values also appear in literature (Berger and Sellke, 1987; Berger and Delampady, 1987).

The paper is organized as follows. A modified p-value is proposed in Section 2. In Section 3, the proposed method is evaluated from a Bayesian point of view and criteria in literature. Further, the modified p-value is shown to be an admissible estimator of the indicator function of Θ_0, while the usual p-value is shown to be an inadmissible estimator of this indicator function. In Section 4, the proposed method is illustrated by a real data example. Simulation studies comparing between the usual p-value and the modified p-value are presented. Section 5 discusses the p-values from the point of view of minimizing the sum of type I and type II errors.

2. Modified *p*-value

In this section, a modified p-value is proposed for testing

$$H_0 : \theta \in \Theta_0 = (b, \theta_0] \quad \text{versus} \quad H_1 : \theta \in \Theta_1 = (\theta_0, a) \tag{2}$$

when $\Theta_0 \cup \Theta_1$ is a restricted parameter space (b, a). Let S denote the parameter space $\Theta_0 \cup \Theta_1$. The p-value derived from the uniformly most powerful test based on the observation x is

$$P_{\theta_0}(X \geqslant x). \tag{3}$$

We propose the modified p-value

$$r_{\theta_0}(x) = \frac{P_{\theta_0}(X \geqslant x) - \min_{\theta \in S} P_\theta(X \geqslant x)}{\max_{\theta \in S} P_\theta(X \geqslant x) - \min_{\theta \in S} P_\theta(X \geqslant x)}$$

to replace (3). For a fixed x, the range of $r_{\theta_0}(x)$ is $(0, 1)$ for $\theta_0 \in S$. When the parameter space is the natural parameter space, $r_{\theta_0}(x)$ is equal to the usual p-value because $\max_{\theta \in S} r_\theta(x) = 1$ and $\min_{\theta \in S} r_\theta(x) = 0$, which leads to $r_{\theta_0}(x) = P_{\theta_0}(X \geqslant x)$.

So, $r_{\theta_0}(x)$ is an x-dependent transformation of the usual p-value (3). The reason for the construction of r_{θ_0} is as follows. For a fixed x, (3) is not greater than $\max_{\theta \in S} P_\theta(X \geqslant x)$ and is not less than $\min_{\theta \in S} P_\theta(X \geqslant x)$. That is, for a fixed x, the range of (3) is $(\min_{\theta \in S} P_\theta(X \geqslant x), \max_{\theta \in S} P_\theta(X \geqslant x))$ when θ_0 belongs to S. Note that if the parameter space is the natural parameter space, the range of the usual p-value is $(0, 1)$ for θ belonging to $(-\infty, \infty)$. When p-value is used as evidence against the null hypothesis, the decision rule for rejection of the null hypothesis is to compare the p-value with the specified level α, which is usually 0.01 or 0.05. If the p-value is less than α, then the null hypothesis is rejected. When the parameter space is restricted, it would be unreasonable to use the same testing level because the usual p-value is greater than $\min_{\theta \in S} P_\theta(X \geqslant x)$ for each x. Thus, a feasible way is to make a transformation of the p-value such that $\min_{\theta \in S} P_\theta(X \geqslant x)$ is not included in a measure of evidence against the null hypothesis and its range is between 0 and 1. Then use the decision rule which rejects the null hypothesis if this transformation of the p-value is less than the level α. From this point of view, the modified p-value $r_{\theta_0}(x)$ is a reasonable transformation of the usual p-value.

For another good property of the modified p-value, consider the extreme case of testing (2) when the parameter space is (b, ∞) and $\theta_0 = b$. In this case, the null hypothesis should obviously be rejected for any x because $b \leqslant \theta$ for all $\theta \in S$. However, the probability that the usual p-value is greater than any specified type I error is positive for all x which indicates that the usual p-value cannot always reject the null hypothesis. However, the modified p-value, which is zero, rejects the null hypothesis for all observations.

3. The Bayesian approach

In this section the modified p-value $r_{\theta_0}(x)$ and the usual p-value are examined by a criterion proposed in Hwang et al. (1992). Hwang et al. provide a criterion to evaluate p-value by considering the loss function

$$L(r(x),\theta) = (r(x) - I(\theta \in \Theta_0))^2, \tag{4}$$

where $r(x)$ denotes a p-value and

$$I(\theta \in \Theta_0) = \begin{cases} 1 & \text{if } \theta \in \Theta_0, \\ 0 & \text{if otherwise.} \end{cases} \tag{5}$$

If a p-value is a good estimator of (5) under the squared error loss, then the p-value is recommended to be a measure of evidence against the null hypothesis. Under the loss function (4), the usual p-value is demonstrated to be an admissible estimator of (5) in the one-sided testing problem for some exponential families with the natural parameter space in Hwang et al. (1992). However, when the parameter spaces are restricted, Woodroofe and Wang (2000) show that the usual p-value is inadmissible under the loss function (4) for Poisson distributions. This discovery reflects the invalidity of using the p-value as an indication against H_0 when the parameter space is restricted. In this paper, the usual p-value is shown to be inadmissible for normal distributions.

First, we will show that the modified p-value is a Bayes estimator of (5), from which the admissibility of the modified p-value is obtained. Note that a Bayes estimator of (5) with respect to the prior $\pi(\theta)$ under the loss function (4) has the form

$$\eta(x) = \int_{\theta\in\Theta_0} e^{-(x-\theta)^2/2}\pi(\theta)\,d\theta \bigg/ \left(\int_{\theta\in\Theta_0} e^{-(x-\theta)^2/2}\pi(\theta)\,d\theta + \int_{\theta\in\Theta_1} e^{-(x-\theta)^2/2}\pi(\theta)\,d\theta\right). \tag{6}$$

Theorem 1. *$r_{\theta_0}(x)$ is equal to the Bayes estimator of $I(\theta \in \Theta_0)$ with respect to prior $\pi(\theta) = I_{(\Theta_0\cup\Theta_1)}(\theta)$, where $I_{(\Theta_0\cup\Theta_1)}(\theta)$ denotes the indicator function of $\Theta_0 \cup \Theta_1$.*

Proof. First, assume that the parameter space is (b,∞). Then Θ_0 in (1) is (b,θ_0). The Bayes estimator of $I(\theta \in \Theta_0)$ with respect to $\pi(\theta) = I_{(b,\infty)}(\theta)$ is

$$\int_b^{\theta_0} e^{-(x-\theta)^2/2}\,d\theta \bigg/ \int_b^{\infty} e^{-(x-\theta)^2/2}\,d\theta. \tag{7}$$

Note that $f(t) = e^{-t^2/2}$ is symmetric around zero. Thus (7) is equal to

$$\begin{aligned}\int_{b-x}^{\theta_0-x} f(t)\,dt \bigg/ \int_{b-x}^{\infty} f(t)\,dt &= \left(\int_{-\infty}^{x-b} f(t)\,dt - \int_{-\infty}^{x-\theta_0} f(t)\,dt\right) \bigg/ \int_{-\infty}^{x-b} f(t)\,dt \\ &= (P_b(X\leqslant x) - P_{\theta_0}(X\leqslant x))/P_b(X\leqslant x) \\ &= (P_{\theta_0}(X\geqslant x) - P_b(X\geqslant x))/(1 - P_b(X\geqslant x)),\end{aligned}$$

which is $r_{\theta_0}(x)$. By a similar argument, when the parameter space is one of the other two kinds of bounded parameter space and the natural parameter space, the results still hold.

Remark 1. To avoid confusion, we use two indicator function notations throughout this paper. Besides the indicator function (5), $I_{(\Theta_0\cup\Theta_1)}(\theta)$ denotes the Lebesque measure restricted to $\Theta_0 \cup \Theta_1$ used as the prior density of θ.

For proving the admissibility of $r_{\theta_0}(x)$, we can apply the result of Theorem 3.3 of Hwang et al. (1992). Note that although this result is for natural parameter space, it can be applied to a restricted parameter space because the tightness property in the proof also holds for a restricted parameter space. Since $r_{\theta_0}(x)$ is a Bayes estimator of $I(\theta \in \Theta_0)$, by a similar argument as Theorem 4.1 in Hwang et al. (1992), we have the following proposition:

Proposition 1. *The Bayes estimator* (6) *is admissible for estimating* (5) *under the loss function* (4).

From Proposition 1, the modified p-value is an admissible estimator for $I(\theta \in \Theta_0)$ for a restricted parameter space. However, the usual p-value is shown to be an inadmissible estimator for $I(\theta \in \Theta_0)$ in Theorem 2.

Theorem 2. *Let X be a normal random variable with distribution* $N(\theta, 1)$. *Assume the parameter space has a lower bound b, and* $\Theta_0 = [b, \theta_0)$. *Then the usual p-value is an inadmissible estimator of* (5) *under the loss function* (4).

Theorem 2 is an extension to the normal distribution of Theorem 1 of Woodroofe and Wang (2000). The proof is in the Appendix. The other restricted parameter space cases follow by a similar argument.

The following two propositions show the relationship between the modified p-value and the usual p-value.

Proposition 2.

1. *Assume the parameter space is* (b, ∞) *then* $r_{\theta_0}(x)$ *is less than or equal to* $P_{\theta_0}(X \geqslant x)$ *for all x and* θ_0.
2. *Assume the parameter space is* $(-\infty, a)$ *then* $r_{\theta_0}(x)$ *is greater than or equal to* $P_{\theta_0}(X \geqslant x)$ *for all x and* θ_0.

Proof. When the parameter space is (b, ∞),

$$r_{\theta_0}(x) - P_{\theta_0}(X \geqslant x) = P_b(X \geqslant x)(P_{\theta_0}(X \geqslant x) - 1)/(1 - P_b(X \geqslant x)),$$

which is less than or equal to zero.

When the parameter space is $(-\infty, a)$, $r_{\theta_0}(x)$ is equal to $P_{\theta_0}(X \geqslant x)/P_a(X \geqslant x)$, which is greater than or equal to $P_{\theta_0}(X \geqslant x)$. □

When the parameter space is (b, a), by numerical calculations, $r_{\theta_0}(x)$ is less than or equal to $P_{\theta_0}(X \geqslant x)$ for $x \leqslant \mu$, and greater than $P_{\theta_0}(X \geqslant x)$ for $x > \mu$, where μ is a constant.

4. Examples

The proposed methodology can be very useful in real applications. It is illustrated by a real-data example about testing the retirement age of civil servants. The data are the retirement ages of civil servants working for Taiwan government. The retirement payment depends on the retirement age. There are two kinds of retirement payment system for these civil servants: (1) A civil servant can continue receiving almost 85% of his salary every month after retirement if he retires after age 50; (2) A civil servant can have a lump sum payment when he retires. Note that if a civil servant retires after 50, he can choose either the first or the second systems; however, if he retires before 50, he can only choose the second system. Most civil servants tend to choose the first system because the total amount of money paid out in the first system is higher than in the second one. But still some people will choose to retire before 50 for personal reasons. According to the law, the retirement age cannot exceed age 65. Therefore, most civil servants' retirement ages are between 50 and 65. Now we are interested in estimating the average retirement age, and can expect that the average age is between 50 and 65. Thus, in this example, we can choose 50 as a lower bound and 65 as an upper bound for the parameter space of the average age.

Example 1. We collected the data of civil servants' retirement ages from government records. There were a total of 9055 civil servants retiring in 2002. In this example, the population is the whole data set, i.e. 9055 civil servants' retirement ages. The mean of these ages is 53.31. The variance is 81.45. Suppose that the researchers do not have the whole data set. A sample of size n is chosen from the data, and they will use the sample to test

$$H_0 : \theta \leqslant \theta_0 \quad \text{versus} \quad H_1 : \theta > \theta_0, \tag{8}$$

where θ is the mean of the data and we assume that retirement age follows a normal distribution $N(\theta, \sigma^2)$. As we mentioned above, it is reasonable to assume that the parameter space has a lower bound 50 and an upper bound 65.

Assume that we have a sample of five ages: 47, 52, 53, 56, 62, whose mean is 54. Assume that θ_0 in (8) is 51. Then the usual p-value for testing (8) is

$$P(Z > (54 - 51)/\sqrt{81.45/5}) = 0.2287,$$

Table 1
Fractions of p-values and of modified p-values less than 0.05 based on 1000 replicates

θ_0	H_0 true or false	n	p-value $\leqslant 0.05$	Modified p-value $\leqslant 0.05$
50	False	5	0.214	1
52	False	5	0.074	0.117
53	False	5	0.04	0.048
54	True	5	0.019	0.019
56	True	5	0.002	0
58	True	5	0	0

Table 2
Fractions of p-values and of modified p-values less than 0.1 based on 1000 replicates

θ_0	H_0 true or false	n	p-value $\leqslant 0.1$	Modified p-value $\leqslant 0.1$
50	False	5	0.354	1
52	False	5	0.188	0.273
53	False	5	0.115	0.141
54	True	5	0.061	0.061
56	True	5	0.011	0.009
58	True	5	0.001	0

Table 3
Fractions of p-values and of modified p-values less than 0.1 based on 1000 replicates

$\theta_1 - \theta_2$	H_0 true or false	n	p-value $\leqslant 0.1$	Modified p-value $\leqslant 0.1$
50	False	5	0.347	1
52	False	5	0.166	0.206
53	False	5	0.121	0.07
54	True	5	0.05	0.02
56	True	5	0.018	0

where Z is the standard normal random variable. Before calculating the modified p-value, we need to derive the values of $\max_{\theta \in S} P_\theta(X \geqslant x)$ and $\min_{\theta \in S} P_\theta(X \geqslant x)$. Here S is (50, 65). The value of $\max_{\theta \in S} P_\theta(X \geqslant x)$ happens at $\theta = 65$, which is $P(Z > (54 - 65)/\sqrt{81.45/5}) = 0.9968$. The value $\min_{\theta \in S} P_\theta(X \geqslant x)$ happens at $\theta = 50$, which is $P(Z > (54 - 50)/\sqrt{81.45/5}) = 0.1608$. The modified p-value is $(0.2287 - 0.1608)/(0.9968 - 0.1608) = 0.0812$. In this case, (8) can be rejected by the modified p-value, but not the usual p-value if the significant level is 0.1. It indicates that the modified p-value is not so conservative as the usual p-value. Simulation results are conducted to compare the two p-values. Tables 1 and 2 show the number of times/1000 they are less than 0.1 and 0.05, respectively, in 1000 samples corresponding to several values of $\theta_0 \in S$ and $n = 5$.

From Tables 1 and 2, when θ_0 is less than 53.31, which means H_0 is false, the number of p-value less than 0.05 is less than that of the modified p-value. When θ_0 is larger than 53.31, we have different results. They reveal that the modified p-value is better than the p-value whether H_0 is true or not.

Example 2. The upper bound in Example 1 is somewhat conservative. A more precise bound can be given from empirical experience. We can collect the data of civil servants' retirement ages of the other two years. The means of the other two years are 55.54, 56.33. Thus, from this empirical knowledge, we can give a more precise upper bound 60. We take the upper bound 60 and the lower bound 50, the results are listed in Table 3.

From Table 3, it is seen that the modified p-value is better than the usual p-value, as well as the results of Example 1. From these simulation results, the performance of the modified p-value which sufficiently utilize the bound information is better than the usual p-value. In real applications, if we have bound information, the modified p-value can certainly provide a more accurate analysis.

5. Minimization of the sum of type I error and type II error

In this section, we will compare the two p-values from the point of view of minimizing the sum of type I and type II errors for simple hypothesis testing. The extension of the simple hypotheses testing to the compositive hypotheses testing is still under investigation.

Consider a simple hypothesis versus a simple alternative hypothesis

$$H_0 : \theta = \theta_0 \quad \text{versus} \quad H_1 : \theta = \theta_1 \quad (\theta_1 > \theta_0). \tag{9}$$

If $X = \theta_0$, then the p-value of the UMP test is $P_{\theta_0}(X > \theta_0) = 0.5$, which does not favor H_0 or H_1. However, when $\theta_1 - \theta_0$ is large, it leads to an unreasonable result because H_0 should not be rejected if $X = \theta_0$. The Bayes estimator, $\eta(x) = f_{\theta_0}(x)/(f_{\theta_0}(x) + f_{\theta_1}(x))$, proposed in this paper with prior $\pi(\theta) = I_{(\Theta_0 \cup \Theta_1)}(\theta)$, is more practical because it is equal to 0.5 when $x = (\theta_0 + \theta_1)/2$ and larger than 0.5 when $x < (\theta_0 + \theta_1)/2$. Thus, it makes sense to suggest rejecting H_0 if

$$\eta(x) < k. \tag{10}$$

If k is 0.5, then from the testing point of view, the test with rejection region $\{x : \eta(x) < k\}$ has the optimal property of minimizing the sum of the two types of errors.

Proposition 3. *Let the random variable X have density function $f_\theta(x)$ with monotone likelihood ratio in x. Then, for testing* (9), *a rejection region which minimizes the sum of type I error and type* II *error is of the form* $\{x : x \geqslant c^*\}$, *where c^* satisfies $f_{\theta_0}(c^*) = f_{\theta_1}(c^*)$.*

Proof. From the Neyman–Pearson lemma, the rejection region such that the sum of type I error and type II error is minimal should be of the form $\{x : x \geqslant c\}$, where c is some positive constant.

Let the rejection region be $S_c = \{x : x \geqslant c\}$. The sum of type I error and type II error is

$$e(c) = \int_c^\infty f_{\theta_0}(x)\,\mathrm{d}x + \int_{-\infty}^c f_{\theta_1}(x)\,\mathrm{d}x.$$

The derivative of $e(c)$ is $(\partial/\partial c)e(c) = -f_{\theta_0}(c) + f_{\theta_1}(c)$. The value c^* such that $e(c)$ is minimal satisfies $(\partial/\partial c)e(c) = 0$. Thus c^* satisfies $f_{\theta_0}(c^*) = f_{\theta_1}(c^*)$. □

For the normal distribution $\mathrm{N}(\theta, 1)$, the value of c satisfying $f_{\theta_0}(c) = f_{\theta_1}(c)$ is $(\theta_0 + \theta_1)/2$. Thus, the rejection region such that the sum of type I and type II errors is minimal is $\{x : x > (\theta_0 + \theta_1)/2\}$. It is reasonable to assume $k = 0.5$ in (10), then the rejection region $\{x : \eta(x) < 0.5\}$ derived from $\eta(x)$ is equal to $\{x : x > (\theta_0 + \theta_1)/2\}$. However, the rejection region $\{x : P(X \geqslant x) < 0.5\}$ derived from the usual p-value is equal to $\{x : x > \theta_0\}$, which is not the rejection region of minimizing the sum of type I and type II errors. It reveals another advantage for $\eta(x)$, but not for the p-value. In fact, for any distribution, the value c satisfying $f_{\theta_0}(c) = f_{\theta_1}(c)$ is the value x for which the Bayes estimator $f_{\theta_0}(x)/(f_{\theta_0}(x) + f_{\theta_1}(x))$ is equal to 0.5. Therefore, from the perspective of minimizing both testing errors, we can see that $\eta(x)$ is more appropriate as a measure evidence against the null hypothesis.

Appendix

Proof of Theorem 2. Note that Theorem 3.3 of Hwang et al. (1992) is also valid for a restricted parameter space. The necessary and sufficient condition of an estimator $\eta(x)$ being admissible given by Theorem 3.3 of Hwang et al. (1992) is that there exists priors $\pi_0(\theta)$ on (b, θ_0) and $\pi_1(\theta)$ on (θ_0, ∞) such that $\eta(x)$ can be expressed as

$$\frac{\int_b^{\theta_0} \mathrm{e}^{(x-\theta)^2/2}\pi_0\{\mathrm{d}\theta\}}{\int_b^{\theta_0} \mathrm{e}^{(x-\theta)^2/2}\pi_0\{\mathrm{d}\theta\} + \int_{\theta_0}^{\infty} \mathrm{e}^{(x-\theta)^2/2}\pi_1\{\mathrm{d}\theta\}} \tag{11}$$

and $\pi = \pi_0 + \pi_1$ should satisfy $\int_{\Theta_0 \cup \Theta_1} e^{(x-\theta)^2/2}\pi\{d\theta\} < \infty$. By using (11), it will be shown that the usual p-value $P_{\theta_0}(X \geqslant x)$ is inadmissible. Assume that $P_{\theta_0}(X \geqslant x)$ is admissible, by (11), we have

$$P_{\theta_0}(X \leqslant x) = \frac{\int_{\theta_0}^{\infty} e^{(x-\theta)^2/2}\pi_1\{d\theta\}}{\int_b^{\theta_0} e^{(x-\theta)^2/2}\pi_0\{d\theta\} + \int_{\theta_0}^{\infty} e^{(x-\theta)^2/2}\pi_1\{d\theta\}}.$$

Thus

$$P_{\theta_0}(X \leqslant x)\int_b^{\infty} e^{(x-\theta)^2/2}\pi\{d\theta\} = \int_{\theta_0}^{\infty} e^{(x-\theta)^2/2}\pi\{d\theta\}. \tag{12}$$

Note that

$$P_{\theta_0}(X \leqslant x) = \int_{-\infty}^{x} e^{(t-\theta_0)^2/2}\, dt = \int_{\theta_0}^{\infty} e^{(t-x)^2/2}\, dt = \int_{\theta_0}^{\infty} e^{tx} e^{-(x^2+t^2)/2}\, dt,$$

which implies $e^{x^2/2} = \int_{-\infty}^{\infty} e^{tx}\, e^{-t^2/2}\, dt$. Let U, Y and W be independent random variables such that $Y = 1$ or 0 with probabilities $\int_{\theta_0}^{\infty} e^{-t^2/2}\, dt$ and $\int_{-\infty}^{\theta_0} e^{-t^2/2}\, dt$, U has density function $e^{-u^2/2}$ and $W = YW_0 + (1-Y)W_1$, where W_0 and W_1 are random variables with density functions $e^{-w^2/2}\pi_0(w)$ and $e^{-w^2/2}\pi_1(w)$, respectively. Then (12) can be rewritten as

$$\int_{\theta_0}^{\infty} e^{xu}\, e^{-u^2/2}\, du \int_b^{\infty} e^{xw}\, e^{-w^2/2}\pi(w)\, dw = \int_{-\infty}^{\infty} e^{xu}\, e^{-u^2/2}\, du \int_{\theta_0}^{\infty} e^{xw}\, e^{-w^2/2}\pi_0(w)\, dw.$$

From the above equation and $P(Y=1) = P(U \geqslant \theta_0)$, we have

$$E(e^{X(U+W)} | U \geqslant \theta_0) = E(e^{X(U+W)} | Y = 1) \tag{13}$$

for all x.

According to moment generating function property, if the moment generating function of a distribution exists in a neighborhood of 0, then the distribution is determined. See, for example, Billingsley (1995, p. 390). Since the moment generating functions of two conditional distributions are the same for all x, the two conditional distributions are the same, which means that:

$$P(U + W \leqslant z | U \geqslant \theta_0) = P(U + W \leqslant z | Y = 1) \tag{14}$$

for all z.

But, when $z < b + \theta_0$, the left-hand side of (14) is 0 since $U \geqslant \theta_0$ and $W \geqslant b$. However, the right-hand side of (14) is always positive. This contradiction is due to the assumption of the p-value being admissible. □

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