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Improved estimation of accuracy in simple hypothesis versus simple alternative testing

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Abstract

In the hypothesis testing problem, a most common used evidence against the null hypothesis is the *p*-value. Although there have been many Bayesian criticisms leveled at *p*-value, Hwang et al. (Ann. Statist. 20 (1992), 490) show the adequacy of using *p*-value as evidence against the null hypothesis by considering testing as an estimation problem. However, when the parameter space is not the natural space, Woodroofe and Wang (Ann. Statist. 28 (2000) 1561) show that the usual *p*-value derived by the N–P test is not appropriate to be the evidence against the null hypothesis for the Poisson distribution from an estimation point of view and provide a modified *p*-value. Although this modified *p*-value is admissible, it is not the admissible estimator which can dominate the usual *p*-value. In this paper, we concentrate on the simple hypothesis versus simple alternative hypothesis testing problem. Admissible estimators which dominate the usual *p*-value are provided.
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1. Introduction

P-value is a well-used measure of evidence against the null hypothesis in hypothesis testing. Although there are many Bayesian and paradox criticisms leveled at *p*-value (e.g. [1–3,7]) some good properties of *p*-value are demonstrated in e.g. (Refs. [5,6,9]). In this paper, we will focus on the complete class of decision rules in the terminology of Hwang et al. [6], which demonstrate some interesting properties

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of p-value by treating the hypothesis testing problem as an estimation problem rather than a decision making problem. In hypothesis testing, assume that the null and alternative hypotheses are

$$\mathrm{H}_0\colon\ \theta\in\Theta_0 \quad \text{versus} \quad \mathrm{H}_1\colon\ \theta\in\Theta_1,$$

where Θ_0 and Θ_1 are two subsets of the natural parameter space Ω and $\Theta_0 \bigcap \Theta_1 = \phi$. Let $I(\theta\in\Theta_0)$ denote the indicator function. Schaarfsma et al. [9] and Hwang et al. [6] suggest that the truth or falsity of a statistical hypothesis H_0 can be discussed by estimating the indicator function $I(\theta\in\Theta_0)$ with squared error loss. Consider the squared error loss function

$$L(\theta, r) = E(r(x) - I(\theta\in\Theta_0))^2, \tag{1}$$

where $r(x)$ denotes an estimator for $I(\theta\in\Theta_0)$. In general, the p-value derived by a reasonable test is a sensible estimator for $I(\theta\in\Theta_0)$ because H_0 is rejected $(I(\theta\in\Theta_0) = 0)$ or accepted $(I(\theta\in\Theta_0) = 1)$ when p-value is too small or too large.

Hwang et al. [6] established some necessary and sufficient conditions for the complete class in the one-sided testing and two-sided testing problems when $\Theta_1 = \Theta_0^c$. The p-value is shown to be admissible in the normal, binomial, and Poisson cases for the one-sided hypothesis testing problem in their paper. But they stated the case $\Theta_1 \neq \Theta_0^c$ was not directly dealt with in their paper although some results can be extended to this case. The parameter space considered in their paper is the natural space including all possible values of parameter. Woodroofe and Wang [11] revealed a controversial concept about the admissibility of the p-value when the parameter space is restricted to some subset of the natural parameter space. It is shown that the usual p-value is inadmissible under the loss function (1) for the Poisson distribution if the parameter space is a strict subset of the natural space. This inadmissibility result also discloses the fact that the usual p-value might not be admissible when the parameter space is not the natural space in other exponential distributions. A modified p-value conditioning on an ancillary statistic is provided in [8,7]. This modified p-value is an admissible estimator for $I(\theta\in\Theta_0)$, however, it is not an admissible estimator which can dominate the usual p-value. Finding an estimator dominating the usual p-value for this case is a difficult problem although we know that better estimators exist. In this paper, we concentrate on the case that Θ_0 and Θ_1 contain one point, respectively, which is a situation of $\Theta_1 \neq \Theta_0^c$ or a case that the parameter space only contains two points. The admissibility results of Hwang et al. [6] can not extend to this simple hypothesis versus simple alternative hypothesis testing problem. A sufficient and necessary condition for an admissible estimator and improved estimators are provided in Section 2. The simple hypothesis versus simple alternative testing is adapted in many practical circumstances: exampling products produced from machine A or machine B, and testing if the fuses produced by a new process average 100 h service life more than that from the old process. Moreover, substantial improvement of the improved estimator is present in Section 3.

Beside simple hypothesis versus simple hypothesis testing, composite hypotheses testing is related to this problem. The results in this paper can not apply to composite testing. The one-sided testing problem for Poisson distribution has been considered by Woodroofe and Wang [11], and the other one-sided testing problems for location families have been discussed in Wang [10].

2. Admissible *p*-value

In this section, we focus on the simple hypothesis versus simple alternative hypothesis testing problem for the exponential family. Without loss of generality, let X be the random variable with density function $k(x)c(\theta)e^{T(x)\theta}$. Then $T(x)$ is a sufficient statistic of θ based on X and assume that the density function of $T(x)$ is

$$f_\theta(t) = h(t)c(\theta)e^{t\theta}.$$

Consider the testing problem of the hypotheses

$$\mathrm{H}_0\colon\ \theta = \theta_0 \quad \text{versus} \quad \mathrm{H}_1\colon\ \theta = \theta_1.$$

We focus on estimating

$$I(\theta = \theta_0) \tag{2}$$

in this paper. An estimator $r(t)$ is admissible for estimating (2), if there does not exist another estimator $r'(t)$ such that

$$E_\theta(r(t) - I(\theta = \theta_0))^2 \geqslant E_\theta(r'(t) - I(\theta = \theta_0))^2 \quad \text{for} \quad \theta = \theta_0 \quad \text{and} \quad \theta = \theta_1$$

and the strict inequality holds for at least one θ. Theorem 1 will give a necessary and sufficient condition of an admissible estimator for estimating (2). By applying the necessary and sufficient condition in Theorem 1, it is shown that the usual *p*-value is inadmissible in Theorem 2. The admissible estimators which dominate the usual *p*-value are provided in Theorem 3.

Theorem 1. *For estimating* (2), *assume that* $\theta_0 < \theta_1$. *Let* ϕ *be an admissible estimator under the loss function* (1). *Then*

(i) ϕ *is a nonincreasing function and there exists a set* $[t_1, t_2]$ *such that* $0 < \phi(t) < 1$ *for all* $t_1 < t < t_2$, $\phi(t) = 1$ *for* $t \leqslant t_1$ *and* $\phi(t) = 0$ *for* $t \geqslant t_2$.
(ii) *There exists a positive constant* m *such that*

$$\phi(t) = \frac{m f_{\theta_0}(t)}{m f_{\theta_0}(t) + f_{\theta_1}(t)} \quad \textit{for all} \quad t_1 < t < t_2.$$

Proof. The admissibility criterion considered in this paper only involves two points $\{\theta_0, \theta_1\}$. Hence, without loss of validity, we can assume that the parameter space

$\Omega = \Omega_0 = \{\theta_0, \theta_1\}$. Thus, according to Theorem 4.14 of Brown [4], if $\phi(t)$ is an admissible estimator for $I(\theta = \theta_0)$, then there exists a sequence π_i of prior distribution supported on θ_0 and θ_1 such that

$$\delta_{\pi_i}(t) \to \phi(t) \quad a.e.,$$

where δ_{π_i} denotes the Bayes estimator for π_i. Hence there exists a sequence (π_{i1}, π_{i2}) of prior distributions such that

$$\begin{aligned}\phi(t) &= \lim_{i\to\infty} \frac{f_{\theta_0}(t)\pi_{i1}(\theta_0)}{f_{\theta_0}(t)\pi_{i1}(\theta_0) + f_{\theta_1}(t)\pi_{i2}(\theta_1)} \\ &= \lim_{i\to\infty} \frac{c(\theta_0)e^{\theta_0 t}\pi_{i1}(\theta_0)}{c(\theta_0)e^{\theta_0 t}\pi_{i1}(\theta_0) + c(\theta_1)e^{\theta_1 t}\pi_{i2}(\theta_1)} \\ &= \lim_{i\to\infty} \frac{\pi_{i1}(\theta_0)}{\pi_{i1}(\theta_0) + c(\theta_1)/c(\theta_0)e^{(\theta_1-\theta_0)t}\pi_{i2}(\theta_1)}. \end{aligned} \tag{3}$$

By Eq. (3), $\phi(t)$ is a nonincreasing function because $\theta_1 - \theta_0 > 0$. Hence there exists an interval $[t_1, t_2]$ such that $\phi(t) = 1$ for $t \leqslant t_1$, $\phi(t) = 0$ for $t \geqslant t_2$ and $0 < \phi(t) < 1$ for $t_1 < t < t_2$. Rewrite the right-hand side of (3) as

$$\begin{aligned}&\lim_{i\to\infty} \frac{f_{\theta_0}(t)\frac{\pi_{i1}(\theta_0)}{\pi_{i2}(\theta_1)}}{f_{\theta_0}(t)\frac{\pi_{i1}(\theta_0)}{\pi_{i2}(\theta_1)} + f_{\theta_1}(t)} \\ &\quad = \frac{f_{\theta_0}(t)\lim_{i\to\infty}\frac{\pi_{i1}(\theta_0)}{\pi_{i2}(\theta_1)}}{f_{\theta_0}(t)\lim\limits_{i\to\infty}\frac{\pi_{i1}(\theta_0)}{\pi_{i2}(\theta_1)} + f_{\theta_1}(t)}.\end{aligned}$$

$\lim_{i\to\infty}\frac{\pi_{i1}(\theta_0)}{\pi_{i2}(\theta_1)}$ exists and is greater than zero because $0 < \phi(t) < 1$. Therefore, $m = \lim_{i\to\infty}\frac{\pi_{i1}(\theta_0)}{\pi_{i2}(\theta_0)}$ and the proof is completed. □

Remark 1. In the other case $\theta_0 > \theta_1$, Theorem 1 is valid if ϕ is changed to a nondecreasing function and $\phi(t) = 0$ for $t \leqslant t_1$, $\phi(t) = 1$ for $t \geqslant t_2$ and $0 < \phi(t) < 1$ for $t_1 \leqslant t \leqslant t_2$.

Theorem 2. *For estimating* (2), *assume that* $\theta_0 < \theta_1$, *the usual p-value* $P_{\theta_0}\left(\frac{f_{\theta_1}(T)}{f_{\theta_0}(T)} \geqslant \frac{f_{\theta_1}(t)}{f_{\theta_0}(t)}\right)$ *derived by the N–P test is inadmissible under the loss function* (1).

Proof. According to Theorem 1, if $P_{\theta_0}\left(\frac{f_{\theta_1}(T)}{f_{\theta_0}(T)} \geqslant \frac{f_{\theta_1}(t)}{f_{\theta_0}(t)}\right)$ is an admissible estimator, then there exists a positive constant m such that

$$P_{\theta_0}\left(\frac{f_{\theta_1}(T)}{f_{\theta_0}(T)} \geqslant \frac{f_{\theta_1}(t)}{f_{\theta_0}(t)}\right) = \frac{mf_{\theta_0}(t)}{mf_{\theta_0}(t) + f_{\theta_1}(t)} \quad \text{for all } t_1 < t < t_2. \tag{4}$$

Modify Eq. (4) such that

$$m = \left(\frac{1}{F\left(Y \leqslant \frac{f_{\theta_1}(t)}{f_{\theta_0}(t)} \right)} - 1 \right) \frac{f_{\theta_1}(t)}{f_{\theta_0}(t)} \quad \text{for all } t_1 < t < t_2, \tag{5}$$

where Y denotes the random variable $\frac{f_{\theta_1}(T)}{f_{\theta_0}(T)}$. Since (5) holds all interior points, $F(y) = \frac{y}{m+y}$ for $0 < F(y) < 1$. Hence the usual p-value is admissible only in the case that the cumulative distribution function is the form $F(y) = \frac{y}{m+y}$ for $0 < F(y) < 1$. The distributions of the exponential families are not the form $F(y) = \frac{y}{m+y}$ for $0 < F(y) < 1$. Thus, the proof is completed. □

Theorem 3. *For estimating* (2) *and* $\theta_0 < \theta_1$, *let* m_0 *and* m_1 *be two positive constants such that*

$$\begin{aligned} E_{\theta_0} & \left[\left(P_{\theta_0} \left(\frac{f_{\theta_1}(T)}{f_{\theta_0}(T)} \geqslant \frac{f_{\theta_1}(t)}{f_{\theta_0}(t)} \right) - \frac{m_0 f_{\theta_0}(t)}{m_0 f_{\theta_0}(t) + f_{\theta_1}(t)} \right) \right. \\ & \times \left. \left(P_{\theta_0} \left(\frac{f_{\theta_1}(T)}{f_{\theta_0}(T)} \geqslant \frac{f_{\theta_1}(t)}{f_{\theta_0}(t)} \right) + \frac{m_0 f_{\theta_0}(t)}{m_0 f_{\theta_0}(t) + f_{\theta_1}(t)} - 2 \right) \right] \\ & = 0 \end{aligned} \tag{6}$$

and

$$\begin{aligned} E_{\theta_1} & \left[\left(P_{\theta_0} \left(\frac{f_{\theta_1}(T)}{f_{\theta_0}(T)} \geqslant \frac{f_{\theta_1}(t)}{f_{\theta_0}(t)} \right) - \frac{m_1 f_{\theta_0}(t)}{m_1 f_{\theta_0}(t) + f_{\theta_1}(t)} \right) \right. \\ & \times \left. \left(P_{\theta_0} \left(\frac{f_{\theta_1}(T)}{f_{\theta_0}(T)} \geqslant \frac{f_{\theta_1}(t)}{f_{\theta_0}(t)} \right) + \frac{m_1 f_{\theta_0}(t)}{m_1 f_{\theta_0}(t) + f_{\theta_1}(t)} \right) \right] \\ & = 0, \end{aligned} \tag{7}$$

then $m_0 < m_1$ *and*

$$r_m(t) = \frac{m f_{\theta_0}(t)}{m f_{\theta_0}(t) + f_{\theta_1}(t)}, \tag{8}$$

where $m_0 < m < m_1$, *is an admissible estimator dominating the usual p-value derived from N–P test under the loss function* (1).

Proof. Let $\theta = \theta_0$, then

$$\begin{aligned} A &= E_{\theta_0} \left(P_{\theta_0} \left(\frac{f_{\theta_1}(T)}{f_{\theta_0}(T)} \geqslant \frac{f_{\theta_1}(t)}{f_{\theta_0}(t)} \right) - I(\theta = \theta_0) \right)^2 - E_{\theta_0} (r_m(t) - I(\theta = \theta_0))^2 \\ &= E_{\theta_0} \left[\left(P_{\theta_0} \left(\frac{f_{\theta_1}(T)}{f_{\theta_0}(T)} \geqslant \frac{f_{\theta_1}(t)}{f_{\theta_0}(t)} \right) - \frac{m f_{\theta_0}(t)}{m f_{\theta_0}(t) + f_{\theta_1}(t)} \right) \right. \\ & \quad \times \left. \left(P_{\theta_0} \left(\frac{f_{\theta_1}(T)}{f_{\theta_0}(T)} \geqslant \frac{f_{\theta_1}(t)}{f_{\theta_0}(t)} \right) + \frac{m f_{\theta_0}(t)}{m f_{\theta_0}(x) + f_{\theta_1}(t)} - 2 \right) \right]. \end{aligned} \tag{9}$$

Let $\theta = \theta_1$, then

$$B = E_{\theta_1}\left(P_{\theta_0}\left(\frac{f_{\theta_1}(T)}{f_{\theta_0}(T)} \geqslant \frac{f_{\theta_1}(t)}{f_{\theta_0}(t)}\right) - I(\theta = \theta_0)\right)^2 - E_{\theta_1}(r_m(t) - I(\theta = \theta_0))^2$$
$$= E_{\theta_1}\left[\left(P_{\theta_0}\left(\frac{f_{\theta_1}(T)}{f_{\theta_0}(T)} \geqslant \frac{f_{\theta_1}(t)}{f_{\theta_0}(t)}\right) - \frac{mf_{\theta_0}(t)}{mf_{\theta_0}(t) + f_{\theta_1}(t)}\right)\right.$$
$$\left. \times \left(P_{\theta_0}\left(\frac{f_{\theta_1}(T)}{f_{\theta_0}(T)} \geqslant \frac{f_{\theta_1}(t)}{f_{\theta_0}(t)}\right) + \frac{mf_{\theta_0}(t)}{mf_{\theta_0}(x) + f_{\theta_1}(t)}\right)\right]. \tag{10}$$

By Theorem 1, an admissible estimator for $I(\theta = \theta_0)$ should be the form of (8) for some constant m. From Theorem 2, the usual p-value $P_{\theta_0}\left(\frac{f_{\theta_1}(T)}{f_{\theta_0}(T)} \geqslant \frac{f_{\theta_1}(t)}{f_{\theta_0}(t)}\right)$ can not be expressed as (8), and thus is inadmissible. The m in the improved estimator (8) should not let A and B be smaller than zero and at least one is greater than zero. In (9), A is negative or positive when $m = 0$ or ∞, respectively. Since A is a continuous function of m, there exits a constant m_0 such that A equals to zero when $m = m_0$ and $A > 0$ for all $m > m_0$. By a similar argument as above, we can deduce that there must exist a constant m_1 such that B equals to zero when $m = m_1$ and $B > 0$ for all $m < m_1$. Thus m_0 must be smaller than m_1, otherwise, admissible estimators dominating the usual p-value do not exist. Thus for $m_0 < m < m_1$, A and B are both positive. The proof is completed. □

Theorem 3 provides all admissible estimators dominating the usual p-value. For a given distribution, we have to calculate m_0 and m_1 first. Tables 1 and 2 will provide m_0 and m_1 for testing the mean, and the variance of a normal distribution.

Moreover, the result in Theorem 3 can extend to other distributions outside of exponential families under some conditions.

Theorem 4. *Assume that $f_\theta(t)$ in Theorem 3 denotes the density function of a distribution $F_\theta(t)$ which is not an exponential family. If m_0 and m_1, satisfying the Eqs. (7) and (8), also satisfy $m_0 < m_1$, then $r_m(t)$, where $m_0 < m < m_1$, is an admissible estimator dominating the usual p-value derived from N–P test under the loss function (1) for the distribution $F_\theta(t)$.*

Proof. By a similar argument as in Theorem 3, we also can create better estimators dominating the usual p-value for other distributions if m_0 and m_1 in (7) and (8) satisfy $m_0 < m_1$. This condition can guarantee that there exists $r_m(t)$, $m_0 < m < m_1$, dominating the usual p-value. For exponential family, the condition $m_0 < m_1$ holds directly from Theorems 2 and 3. □

Theorems 5 and 6 specify some relationship between $|\theta_0 - \theta_1|$ and $m_i,\ i = 1, 2$.

Theorem 5. *In Theorem 3, if x is a normal random variable with an unknown mean θ and variance 1, then m_0 in Theorem 3 only depends on the value $|\theta_0 - \theta_1|$ and m_1 in Theorem 3 only depends on the value $(\theta_0 - \theta_1)$.*

Table 1

$\theta_0-\theta_1$	m_0	m_1	$\theta_0-\theta_1$	m_0	m_1
0.2	0.721399	1.13248	−0.2	0.721399	1.13248
0.4	0.69022	0.952231	−0.4	0.69022	0.952231
0.6	0.640771	0.800931	−0.6	0.640771	0.800931
0.8	0.57662	0.667737	−0.8	0.57662	0.667737
1	0.502397	0.548704	−1	0.502397	0.548705
1.2	0.423315	0.442924	−1.2	0.423315	0.442925
1.4	0.344573	0.350494	−1.4	0.344573	0.350499
1.6	0.270707	0.271543	−1.6	0.270707	0.271548
1.8	0.205109	0.2058	−1.8	0.205109	0.2058
2	0.149785	0.15249	−2	0.149785	0.152489
2.2	0.105369	0.110422	−2.2	0.105369	0.11042
2.4	0.0713745	0.0781199	−2.4	0.0713745	0.0781248
2.6	0.0465377	0.053992	−2.6	0.0465377	0.053991
2.8	0.0291996	0.0364428	−2.8	0.0291996	0.0364382
3	0.0176261	0.0240198	−3	0.0176261	0.0240207
3.2	0.0102344	0.0154559	−3.2	0.0102344	0.0154594
3.4	0.00571505	0.00971459	−3.4	0.00571505	0.00971201
3.6	0.00306888	0.0059594	−3.6	0.00306888	0.00595962
3.8	0.00158449	0.00356744	−3.8	0.00158449	0.00356889
4	0.000786514	0.00208646	−4	0.000786514	0.00208474
4.2	0.000375318	0.00119047	−4.2	0.000375318	0.00119057
4.4	0.000172162	0.000652023	−4.4	0.000172162	0.000662923
4.6	0.0000759095	0.000352465	−4.6	0.0000759095	0.000356785
4.8	0.0000321703	0.000191068	−4.8	0.0000321703	0.000183388
5	0.0000131037	0.0000973375	−5	0.0000131037	0.0000981856

Proof. First consider the case of m_0. The right-hand side of (9) is

$$\int\left(\int_x^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{(s-\theta_0)^2}{2}}\,\mathrm{d}s-\frac{m_0e^{\frac{-(x-\theta_0)^2}{2}}}{m_0e^{\frac{-(x-\theta_0)^2}{2}}+e^{\frac{-(x-\theta_1)^2}{2}}}\right)\times\left(\int_x^\infty \frac{1}{\sqrt{2\pi}}e^{\frac{-(s-\theta_0)^2}{2}}\,\mathrm{d}s+\frac{m_0e^{\frac{-(x-\theta_0)^2}{2}}}{m_0e^{\frac{-(x-\theta_0)^2}{2}}+e^{\frac{-(x-\theta_1)^2}{2}}}-2\right)\frac{1}{\sqrt{2\pi}}e^{\frac{-(x-\theta_0)^2}{2}}\,\mathrm{d}x. \tag{11}$$

Let $(x-\theta_0)=y$, then (11) equals

$$\int\left(\int_y^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{s^2}{2}}\,\mathrm{d}s-\frac{m_0e^{-\frac{y^2}{2}}}{m_0e^{-\frac{y^2}{2}}+e^{-\frac{(y+\theta_0-\theta_1)^2}{2}}}\right)\times\left(\int_y^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{s^2}{2}}\,\mathrm{d}s+\frac{m_0e^{-\frac{y^2}{2}}}{m_0e^{-\frac{y^2}{2}}+e^{-\frac{(y+\theta_0-\theta_1)^2}{2}}}-2\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\,\mathrm{d}y. \tag{12}$$

Table 2

σ_0/σ_1	m_0	m_1	σ_0/σ_1	m_0	m_1
0.25	2.13479	24.1522	1/0.25	0.150515	0.986357
0.5	1.33524	4.74288	1/0.5	0.345573	1.06716
0.75	0.960196	2.23134	1/0.75	0.542187	1.19891
1	0.732051	1.36603	1/1	0.732051	1.36602
1.25	0.58103	0.952992	1/1.25	0.906144	1.54739
1.5	0.476916	0.71823	1/1.5	1.06284	1.72428
1.75	0.401895	0.569064	1/1.75	1.20493	1.89102
2	0.345573	0.466877	1/2	1.33524	2.04744
2.25	0.301843	0.393007	1/2.25	1.45599	2.19458
2.5	0.266958	0.337415	1/2.5	1.56886	2.33362
2.75	0.238509	0.294252	1/2.75	1.6751	2.46561
3	0.214887	0.259889	1/3	1.77568	2.59142
3.25	0.194975	0.231969	1/3.25	1.87139	2.71179
3.5	0.177974	0.208893	1/3.5	1.96282	2.82735
3.75	0.1633	0.189544	1/3.75	2.05049	2.93859
4	0.150515	0.173117	1/4	2.13479	3.04595
4.25	0.139283	0.15902	1/4.25	2.21609	3.14979
4.5	0.129343	0.146808	1/4.5	2.29466	3.25043
4.75	0.120491	0.136141	1/4.75	2.37076	3.34813
5	0.112561	0.126755	1/5	2.4446	3.44314
5.25	0.10542	0.11844	1/5.25	2.51636	3.53568
5.5	0.0989612	0.111031	1/5.5	2.5862	3.6259
5.75	0.093093	0.104392	1/5.75	2.65427	3.71397
6	0.087742	0.0984136	1/6	2.7207	3.80004
6.25	0.0828446	0.0930076	1/6.25	2.78559	3.88424
6.5	0.078348	0.0880978	1/6.5	2.84904	3.96668
6.75	0.0742069	0.0836216	1/6.75	2.91115	4.04747
7	0.0703827	0.0795265	1/7	2.97199	4.1267
7.25	0.0668427	0.0757679	1/7.25	3.03164	4.20446
7.5	0.0635572	0.0723076	1/7.5	3.09016	4.28082
7.75	0.0605014	0.069113	1/7.75	3.14761	4.35586
8	0.0576534	0.066156	1/8	3.20406	4.42965
8.25	0.0549939	0.0634121	1/8.25	3.25954	4.50223
8.5	0.0525058	0.0608601	1/8.5	3.31411	4.57368
8.75	0.0501741	0.0584813	1/8.75	3.36782	4.64404
9	0.0479859	0.0562596	1/9	3.42069	4.71335
9.25	0.0459287	0.0541805	1/9.25	3.47277	4.78167
9.5	0.043992	0.0522313	1/9.5	3.52409	4.84903
9.75	0.0421665	0.0504009	1/9.75	3.57468	4.91548
10	0.0404434	0.0486788	1/10	3.62458	4.98105

Expand the $e^{-\frac{(y+\theta_0-\theta_1)^2}{2}}$ in (12) to $e^{-\frac{y^2}{2}} \cdot e^{y(\theta_1-\theta_0)} \cdot e^{-\frac{(\theta_0-\theta_1)^2}{2}}$. Note that y is symmetric about zero. Therefore (12) only depends on $|\theta_0 - \theta_1|$. Thus, m_0 only depends on $|\theta_0 - \theta_1|$. By a similar argument as above, (10) depends on $(\theta_0 - \theta_1)$. Hence, m_1 only depends on $(\theta_0 - \theta_1)$. $\square$

Theorem 6. *In Theorem* 5, m_0 *and* m_1 *go to zero when* $|\theta_0 - \theta_1|$ *goes to infinity.*

The proof of Theorem 6 is in the Appendix.

Theorem 7. *Assume that* x *is a random normal variable with density function* $\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$. *For testing* H_0: $\sigma^2 = \sigma_0^2$ *versus* H_1: $\sigma^2 = \sigma_1^2$, m_0 *and* m_1 *in Theorem* 3 *only depend on* $\frac{\sigma_0^2}{\sigma_1^2}$.

Proof. By an argument similar to that of Theorem 5, Theorem 7 can be proved. □

We will call the improved estimators (8) derived by Theorem 3 as admissible estimators below. In this paper, we will list the upper and lower bounds of m in (8) for testing the mean and the variance of a normal distribution. These bounds are derived by software Mathematica. The calculations of other exponential families can also be deduced straightforwardly.

Table 1 lists the upper and lower bounds of m in (8) for testing the mean of a normal distribution. Note that by Theorem 4, these values only depend on the difference of two means θ_0 and θ_1.

Table 2 lists the two bounds of m in (8) for testing the variance of a normal distribution. From Theorem 5, m_0 and m_1 only rely on the ratio of two variances σ_1^2 and σ_2^2.

3. Improvement of the modified p-values

In this section, substantial improvement of the admissible estimators are revealed. By calculating the two mean squared errors of admissible estimators in which m is chosen to be $\frac{m_0+m_1}{2}$ and the usual p-value, we find the improvement of (8) is significant. Tables 3 and 4 list the ratios of MSE $E\left(r_{\frac{m_0+m_1}{2}}(t) - I(\mathrm{H}_0 \quad true)\right)^2$ and MSE $E\left(P_{\theta_0}\left(\frac{f_{\theta_1}(T)}{f_{\theta_0}(T)} \geqslant \frac{f_{\theta_1}(t)}{f_{\theta_0}(t)}\right) - I(\mathrm{H}_0 \ \ true)\right)^2$ for testing the mean and the variance of a normal distribution.

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Appendix

Table 3

$\theta_0-\theta_1$	θ_0 true	θ_1 true	$\theta_0-\theta_1$	θ_0 true	θ_1 true
0.2	0.799395	0.822032	−0.2	0.799395	0.822032
0.4	0.864423	0.85997	−0.4	0.864423	0.85997
0.6	0.912546	0.896443	−0.6	0.912546	0.896443
0.8	0.947031	0.929039	−0.8	0.947031	0.929039
1	0.970796	0.956286	−1	0.970796	0.956287
1.2	0.98624	0.97726	−1.2	0.986239	0.977261
1.4	0.995236	0.991402	−1.4	0.995232	0.991409
1.6	0.999204	0.998435	−1.6	0.9992	0.998444
1.8	0.999196	0.99833	−1.8	0.999196	0.99833
2	0.995983	0.991281	−2	0.995985	0.991278
2.2	0.99012	0.977715	−2.2	0.990124	0.977706
2.4	0.98198	0.958259	−2.4	0.981968	0.958326
2.6	0.971788	0.933931	−2.6	0.971792	0.933923
2.8	0.959734	0.905531	−2.8	0.959758	0.90547
3	0.945896	0.874193	−3	0.945889	0.874211
3.2	0.93034	0.840971	−3.2	0.930299	0.841079
3.4	0.912965	0.807255	−3.4	0.913013	0.807128
3.6	0.893907	0.773711	−3.6	0.893901	0.773729
3.8	0.873193	0.741182	−3.8	0.873122	0.741376
4	0.850635	0.711007	−4	0.850778	0.710614
4.2	0.826547	0.683068	−4.2	0.826532	0.683108
4.4	0.803757	0.650098	−4.4	0.800957	0.657965
4.6	0.777649	0.625524	−4.6	0.775629	0.631273
4.8	0.745922	0.616928	−4.8	0.752576	0.597624
5	0.719397	0.593435	−5	0.718028	0.597553

Proof of Theorem 6. First, consider the case of m_0, which has to satisfy Eq. (12) $=0$. Note that (12) can be rewritten as

$$\int_{-\infty}^{\infty}\left(1-F(y)-\frac{1}{k(y)}\right)\times\left(1-F(y)+\frac{1}{k(y)}-2\right)\mathrm{d}F(y), \tag{A.1}$$

where $F(y)=\int_{-\infty}^{y}e^{-s^2/2}/\sqrt{2\pi}\,\mathrm{d}s$ and $k(y)=1+e^{y(\theta_1-\theta_0)-(\theta_0-\theta_1)^2/2}/m_0$. Then (A.1) can be rewritten as

$$\begin{aligned}&\int_{-\infty}^{\infty}(-1+F^2(y))\,\mathrm{d}F(y)+\int_{-\infty}^{\infty}\left(\frac{2}{k(y)}-\frac{1}{k^2(y)}\right)\mathrm{d}F(y)\\&\quad=-\frac{2}{3}+\int_{-\infty}^{\infty}\left(\frac{2}{k(y)}-\frac{1}{k^2(y)}\right)\mathrm{d}F(y).\end{aligned}$$

Since m_0 has to satisfy (12) $=0$, thus m_0 has to satisfy

$$\frac{2}{3}=\int_{-\infty}^{\infty}\frac{1}{k(y)}\left(2-\frac{1}{k(y)}\right)\mathrm{d}F(y). \tag{A.2}$$

Table 4

σ_0/σ_1	θ_0 true	θ_1 true	σ_0/σ_1	θ_0 true	θ_1 true
0.25	0.140491	0.806319	1/0.25	0.300839	0.715799
0.5	0.44327	0.7449	1/0.5	0.511514	0.734071
0.75	0.61475	0.759377	1/0.75	0.638949	0.762978
1	0.71453	0.78633	1/1	0.714533	0.786326
1.25	0.777917	0.811322	1/1.25	0.758484	0.802941
1.5	0.821542	0.832229	1/1.5	0.785266	0.814747
1.75	0.853251	0.849719	1/1.75	0.802922	0.823369
2	0.877154	0.864511	1/2	0.815335	0.829872
2.25	0.895642	0.877117	1/2.25	0.824559	0.834928
2.5	0.910207	0.887908	1/2.5	0.831632	0.838964
2.75	0.921841	0.897157	1/2.75	0.837235	0.842255
3	0.931227	0.905072	1/3	0.841782	0.844984
3.25	0.938852	0.911827	1/3.25	0.845542	0.847287
3.5	0.945076	0.917553	1/3.5	0.848702	0.849254
3.75	0.950166	0.922368	1/3.75	0.851394	0.850954
4	0.95433	0.92637	1/4	0.853716	0.852436
4.25	0.957729	0.929637	1/4.25	0.855738	0.853741
4.5	0.960492	0.932239	1/4.5	0.857515	0.854897
4.75	0.962716	0.934247	1/4.75	0.85909	0.855928
5	0.964485	0.93571	1/5	0.860492	0.856856
5.25	0.965871	0.936672	1/5.25	0.861751	0.857693
5.5	0.96692	0.93719	1/5.5	0.862888	0.858452
5.75	0.967687	0.937288	1/5.75	0.863919	0.859143
6	0.968208	0.937006	1/6	0.864857	0.859776
6.25	0.968511	0.936382	1/6.25	0.865715	0.860358
6.5	0.968627	0.935438	1/6.5	0.866504	0.860894
6.75	0.968581	0.934199	1/6.75	0.867229	0.86139
7	0.968392	0.93269	1/7	0.867901	0.86185
7.25	0.968074	0.930936	1/7.25	0.868522	0.862277
7.5	0.967646	0.928952	1/7.5	0.869101	0.862675
7.75	0.96712	0.926756	1/7.75	0.869641	0.863047
8	0.966507	0.924369	1/8	0.870143	0.863397
8.25	0.965816	0.921802	1/8.25	0.870616	0.863723
8.5	0.965057	0.919071	1/8.5	0.871058	0.864031
8.75	0.964238	0.916187	1/8.75	0.871473	0.864323
9	0.963362	0.91317	1/9	0.871865	0.864595
9.25	0.962438	0.910022	1/9.25	0.872235	0.864854
9.5	0.961472	0.906756	1/9.5	0.872584	0.865098
9.75	0.960466	0.903387	1/9.75	0.872915	0.86533
10	0.959427	0.899914	1/10	0.873228	0.865551

Assume that $\theta_1 > \theta_0$, then the right-hand side of (A.2) is greater than

$$\int_{-\infty}^{\infty} \frac{1}{k(y)} \, \mathrm{d}F(y)$$

$$
\begin{aligned}
&\geqslant \int_{-\infty}^{0} \frac{1}{1+\frac{1}{m_0} e^{-\frac{(\theta_0-\theta_1)^2}{2}}} \mathrm{d}F(y) + \int_0^{\infty} \frac{1}{k(y)} \mathrm{d}F(y) \\
&= \frac{\frac{1}{2}}{1+\frac{1}{m_0} e^{-\frac{(\theta_0-\theta_1)^2}{2}}} + \int_0^{\infty} \frac{1}{k(y)} \mathrm{d}F(y),
\end{aligned}
$$

which leads to

$$
m_0 \leqslant e^{-\frac{(\theta_0-\theta_1)^2}{2}} \Bigg/ \left(\frac{1}{2\left(\frac{2}{3} - \int_0^{\infty} \frac{1}{k(y)} \mathrm{d}F(y)\right)} - 1 \right). \tag{A.3}
$$

When $\theta_1 - \theta_0$ goes to infinity, the term $\lim_{\theta_1-\theta_0\to\infty} \int_0^{\infty} 1/k(y)\, \mathrm{d}F(y)$ in (A.3) goes to $\frac{1}{2}$, which leads that the right-hand side of (A.3) is equal to $e^{-(\theta_0-\theta_1)^2/2}/2$. Thus, m_0 goes to zero when $\theta_1 - \theta_0$ goes to infinity. By a similar argument, θ_1 is less than θ_0, we have

$$
m_0 \leqslant e^{-\frac{(\theta_0-\theta_1)^2}{2}} \Bigg/ \left(\frac{1}{2\left(\frac{2}{3} - \int_0^{\infty} \frac{1}{k(y)} \mathrm{d}F(y)\right)} - 1 \right)
$$

which goes to zero as $\theta_0 - \theta_1$ goes to infinity.

For the case of m_1, m_1 has to satisfy the equation

$$
\int_{-\infty}^{\infty} \left(1 - F(y) - \frac{1}{k^*(y)}\right) \times \left(1 - F(y) + \frac{1}{k^*(y)}\right) \mathrm{d}F(y) = 0, \tag{A.4}
$$

where $k^*(y) = 1/(1 + e^{y(\theta_1-\theta_0)-(\theta_0-\theta_1)^2/2}/m_1)$. Then the left-hand side of (A.4) can be rewritten as

$$
\begin{aligned}
&\int_{-\infty}^{\infty} (1 - 2F(y) + F^2(y))\, \mathrm{d}F(y) - \int_{-\infty}^{\infty} \left(\frac{1}{k^*(y)}\right)^2 \mathrm{d}F(y) \\
&= \frac{1}{3} - \int_{-\infty}^{\infty} \left(\frac{1}{k^*(y)}\right)^2 \mathrm{d}F(y)
\end{aligned} \tag{A.5}
$$

Thus, m_1 has to satisfy (A.6) $= 0$. By the fact and Cauchy–Schwarz inequality, we have

$$
\frac{1}{3} \geqslant \int_{-\infty}^{\infty} \frac{1}{\left(1 + \frac{e^{-(\theta_0-\theta_1)^2}}{m_1^2}\right)(1 + e^{2y(\theta_1-\theta_0)})} \mathrm{d}F(y)
$$

which leads to

$$
1 + \frac{e^{-(\theta_0-\theta_1)^2}}{m_1^2} \geqslant 3 \int_{-\infty}^{\infty} \frac{1}{1+e^{2y(\theta_1-\theta_0)}} \mathrm{d}F(y).
$$

By a straightforward calculation, we have

$$m_1^2 \leqslant \frac{e^{-(\theta_0-\theta_1)^2}}{3\int_{-\infty}^{\infty} \frac{1}{1+e^{2y(\theta_1-\theta_0)}}\,\mathrm{d}F(y) - 1}. \tag{A.6}$$

The denominator of the right-hand side of (A.6) can be rewritten as

$$3\int_0^{\infty} \frac{1}{1+e^{2y(\theta_1-\theta_0)}}\,\mathrm{d}F(y) + 3\int_{-\infty}^{0} \frac{1}{1+e^{2y(\theta_1-\theta_0)}}\,\mathrm{d}F(y) - 1,$$

which tends to $3/2 - 1$ as $|\theta_1 - \theta_0|$ goes to infinity. Thus, when $|\theta_1 - \theta_0|$ goes to infinity, m_1 goes to zero. □

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